

# HHO y Exponential Fitting para problemas de convección–difusión singularmente perturbados

Melani Barrios<sup>1</sup> Ariel Lombardi<sup>1 2</sup>  
Cecilia Penessi<sup>1 2</sup>

<sup>1</sup>FCEIA - Universidad Nacional de Rosario

<sup>2</sup>CONICET

UMA 2022 - Sesión 3: “Análisis Numérico y Optimización”

Septiembre 2022



- 1 Problema de convección-difusión *singularmente perturbado*
- 2 Hybrid High-Order Method
- 3 Exponential Fitting
- 4 Implementaciones numéricas

## Problema de convección-difusión

Hallar  $u$  tal que:

$$\begin{cases} -\operatorname{div}(\varepsilon \nabla u - \bar{\beta} u) = f & \text{en } \Omega, \\ u = g & \text{en } \Gamma, \end{cases}$$

## Problema de convección-difusión

Hallar  $u$  tal que:

$$\begin{cases} -\operatorname{div}(\varepsilon \nabla u - \bar{\beta} u) = f & \text{en } \Omega, \\ u = g & \text{en } \Gamma, \end{cases}$$

- $\Omega$  dominio poligonal,  $\Gamma = \partial\Omega$ .
- Término de convección:  $\bar{\beta} = (b_1, b_2)$  campo vectorial con  $b_1, b_2 > \gamma > 0$  en  $\Omega$ .
- $f$  y  $g$  funciones suaves en  $\bar{\Omega}$ .
- Parámetro de difusión:  $\varepsilon \in (0, 1]$

## Problema de convección-difusión

Hallar  $u$  tal que:

$$\begin{cases} -\operatorname{div}(\varepsilon \nabla u - \bar{\beta} u) = f & \text{en } \Omega, \\ u = g & \text{en } \Gamma, \end{cases}$$

- $\Omega$  dominio poligonal,  $\Gamma = \partial\Omega$ .
- Término de convección:  $\bar{\beta} = (b_1, b_2)$  campo vectorial con  $b_1, b_2 > \gamma > 0$  en  $\Omega$ .
- $f$  y  $g$  funciones suaves en  $\bar{\Omega}$ .
- Parámetro de difusión:  $\varepsilon \in (0, 1] \rightarrow \varepsilon \ll 1$

Problema de convección-difusión con convección dominante

## Problema de convección-difusión

Hallar  $u$  tal que:

$$\begin{cases} -\operatorname{div}(\varepsilon \nabla u - \bar{\beta} u) = f & \text{en } \Omega, \\ u = g & \text{en } \Gamma, \end{cases}$$

- $\Omega$  dominio poligonal,  $\Gamma = \partial\Omega$ .
- Término de convección:  $\bar{\beta} = (b_1, b_2)$  campo vectorial con  $b_1, b_2 > \gamma > 0$  en  $\Omega$ .
- $f$  y  $g$  funciones suaves en  $\bar{\Omega}$ .
- Parámetro de difusión:  $\varepsilon \in (0, 1] \rightarrow \varepsilon \ll 1$

Problema de convección-difusión con convección dominante



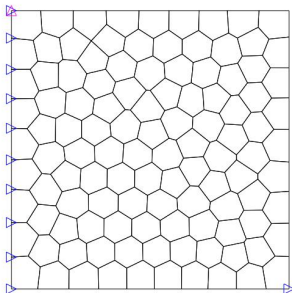
CAPAS LÍMITE

# Tabla de Contenidos

- 1 Problema de convección-difusión *singularmente perturbado*
- 2 Hybrid High–Order Method
- 3 Exponential Fitting
- 4 Implementaciones numéricas

## ALGUNAS CARACTERÍSTICAS

- posibilidad de utilizar mallas poligonales arbitrarias (incluyendo nodos colgantes);
- posibilidad de considerar órdenes polinómicos arbitrarios (incluyendo  $k = 0$ ).



$$\mathcal{M}_h = (\mathcal{T}_h, \mathcal{F}_h)$$



# PROBLEMA MODELO

Hallar  $u : \Omega \rightarrow \mathbb{R}$  tal que

$$\begin{cases} -\operatorname{div}(M\nabla u) = f & \text{en } \Omega, \\ u = 0 & \text{en } \Gamma. \end{cases}$$

# PROBLEMA MODELO

Hallar  $u : \Omega \rightarrow \mathbb{R}$  tal que

$$\begin{cases} -\operatorname{div}(M\nabla u) = f & \text{en } \Omega, \\ u = 0 & \text{en } \Gamma. \end{cases}$$

## Formulación débil clásica

Hallar  $u \in H_0^1(\Omega)$  tal que:

$$a(u, v) = (f, v) \quad \forall v \in H_0^1(\Omega),$$

forma bilineal  $a : H^1(\Omega) \times H^1(\Omega) \rightarrow \mathbb{R}$ ,  $a(u, v) := (M\nabla u, \nabla v)$ .

Suponemos  $M|_T \equiv m_T \in \mathcal{P}_0(\mathcal{T}_h)$ .

## Espacio discreto local:

$$\underline{U}_T^k := \{ \underline{v}_T = (v_T, (v_F)_{F \in \mathcal{F}_T}) : v_T \in \mathbb{P}^k(T), v_F \in \mathbb{P}^k(F) \forall F \in \mathcal{F}_T \}$$

## Espacio discreto local:

$$\underline{U}_T^k := \{ \underline{v}_T = (v_T, (v_F)_{F \in \mathcal{F}_T}) : v_T \in \mathbb{P}^k(T), v_F \in \mathbb{P}^k(F) \forall F \in \mathcal{F}_T \}$$

## Proyección $L^2$

$$\begin{aligned} \pi_X^{0,\ell} : L^1(X) &\rightarrow \mathbb{P}^\ell(X), \\ v &\mapsto \pi_X^{0,\ell} v \quad \text{tal que} \end{aligned}$$

$$\left( \pi_X^{0,\ell} v - v, \omega \right)_X = 0 \quad \forall \omega \in \mathbb{P}_\ell(X) \quad \text{y} \quad \pi_X^{0,\ell} v = \arg \min_{\omega \in \mathbb{P}^\ell(X)} \|\omega - v\|_X^2$$

## Espacio discreto local:

$$\underline{U}_T^k := \{ \underline{v}_T = (v_T, (v_F)_{F \in \mathcal{F}_T}) : v_T \in \mathbb{P}^k(T), v_F \in \mathbb{P}^k(F) \forall F \in \mathcal{F}_T \}$$

## Proyección $L^2$

$$\begin{aligned} \pi_X^{0,\ell} : L^1(X) &\rightarrow \mathbb{P}^\ell(X), \\ v &\mapsto \pi_X^{0,\ell} v \quad \text{tal que} \end{aligned}$$

$$\left( \pi_X^{0,\ell} v - v, \omega \right)_X = 0 \quad \forall \omega \in \mathbb{P}_\ell(X) \quad \text{y} \quad \pi_X^{0,\ell} v = \arg \min_{\omega \in \mathbb{P}^\ell(X)} \|\omega - v\|_X^2$$

## Interpolador local

$$\begin{aligned} \underline{I}_T^k : W^{1,1}(T) &\rightarrow \underline{U}_T^k, \\ v &\mapsto \underline{I}_T^k v := \left( \pi_T^{0,k} v, (\pi_F^{0,k} v)_{F \in \mathcal{F}_T} \right) \end{aligned}$$

## Seminorma de tipo $H^1$

$$\|\underline{v}_T\|_{1,T} := \left( \|\nabla v_T\|_T^2 + |\underline{v}_T|_{1,\partial T}^2 \right)^{\frac{1}{2}}; |\underline{v}_T|_{1,\partial T} := \left( \sum_{F \in \mathcal{F}_T} h_F^{-1} \|v_F - v_T\|_F^2 \right)^{\frac{1}{2}}$$

## Seminorma de tipo $H^1$

$$\|\underline{v}_T\|_{1,T} := \left( \|\nabla v_T\|_T^2 + |\underline{v}_T|_{1,\partial T}^2 \right)^{\frac{1}{2}}; |\underline{v}_T|_{1,\partial T} := \left( \sum_{F \in \mathcal{F}_T} h_F^{-1} \|v_F - v_T\|_F^2 \right)^{\frac{1}{2}}$$

$$\|I_T^k v\|_{1,T} \leq C(d, \rho, k) |v|_{H^1(T)}$$

## Seminorma de tipo $H^1$

$$\|\underline{v}_T\|_{1,T} := (\|\nabla v_T\|_T^2 + |\underline{v}_T|_{1,\partial T}^2)^{\frac{1}{2}}; |\underline{v}_T|_{1,\partial T} := \left( \sum_{F \in \mathcal{F}_T} h_F^{-1} \|v_F - v_T\|_F^2 \right)^{\frac{1}{2}}$$

$$\|\underline{I}_T^k v\|_{1,T} \leq C(d, \rho, k) |v|_{H^1(T)}$$

## Operador de reconstrucción

$$\begin{aligned} \rho_T^{k+1} : \underline{U}_T^k &\rightarrow \mathbb{P}^{k+1}(T), \\ \underline{v}_T &\mapsto \rho_T^{k+1} \underline{v}_T \quad \text{tal que} \end{aligned}$$

$$(\nabla \rho_T^{k+1} \underline{v}_T, \nabla \omega)_T = -(v_T, \delta \omega)_T + \sum_{F \in \mathcal{F}_T} (v_F, \nabla \omega \cdot n_{TF})_F \quad \forall \omega \in \mathbb{P}^{k+1}(T);$$

$$(\rho_T^{k+1} \underline{v}_T - v_T, 1)_T = 0$$



Forma bilineal discreta  $a_T : \underline{U}_T^k \times \underline{U}_T^k \rightarrow \mathbb{R}$ ,

$$a_T(\underline{u}_T, \underline{v}_T) = \underbrace{\left( m_T \nabla p_T^{k+1} \underline{u}_T, \nabla p_T^{k+1} \underline{v}_T \right)}_{\text{Consistencia}} + \underbrace{s_T(\underline{u}_T, \underline{v}_T)}_{\text{Coercitividad}}$$

Forma bilineal discreta  $a_T : \underline{U}_T^k \times \underline{U}_T^k \rightarrow \mathbb{R}$ ,

$$a_T(\underline{u}_T, \underline{v}_T) = \underbrace{\left( m_T \nabla p_T^{k+1} \underline{u}_T, \nabla p_T^{k+1} \underline{v}_T \right)}_{\text{Consistencia}} + \underbrace{s_T(\underline{u}_T, \underline{v}_T)}_{\text{Coercitividad}}$$

→ Estabilización original HHO

$$s_T(\underline{u}_T, \underline{v}_T) := m_T \sum_{F \in \mathcal{F}_T} h_F^{-1} \left( \left( \delta_{TF}^k - \delta_T^k \right) \underline{u}_T, \left( \delta_{TF}^k - \delta_T^k \right) \underline{v}_T \right)_F$$

Forma bilineal discreta  $a_T : \underline{U}_T^k \times \underline{U}_T^k \rightarrow \mathbb{R}$ ,

$$a_T(\underline{u}_T, \underline{v}_T) = \underbrace{\left( m_T \nabla p_T^{k+1} \underline{u}_T, \nabla p_T^{k+1} \underline{v}_T \right)}_{\text{Consistencia}} + \underbrace{s_T(\underline{u}_T, \underline{v}_T)}_{\text{Coercitividad}}$$

→ Estabilización original HHO

$$s_T(\underline{u}_T, \underline{v}_T) := m_T \sum_{F \in \mathcal{F}_T} h_F^{-1} \left( \left( \delta_{TF}^k - \delta_T^k \right) \underline{u}_T, \left( \delta_{TF}^k - \delta_T^k \right) \underline{v}_T \right)_F$$

## Operadores diferencia

$$\begin{aligned} \delta_T^k : \underline{U}_T^k &\rightarrow \mathbb{P}^k(T) & ; \delta_{TF}^k : \underline{U}_T^k &\rightarrow \mathbb{P}^k(F) \\ \underline{v}_T &\mapsto \pi_T^{0,k} (p_T^{k+1} \underline{v}_T - v_T) & \underline{v}_T &\mapsto \pi_F^{0,k} (p_T^{k+1} \underline{v}_T - v_F) \end{aligned}$$

Espacio GLOBAL de incógnitas discretas

$$\underline{U}_h^k := \left\{ \underline{v}_h = ((v_T)_{T \in \mathcal{T}_h}, (v_F)_{F \in \mathcal{F}_h}) : v_T \in \mathbb{P}^k(T) \forall T \in \mathcal{T}_h, \right. \\ \left. v_F \in \mathbb{P}^k(F) \forall F \in \mathcal{F}_h \right\}$$

# Problema discreto

Espacio GLOBAL de incógnitas discretas

$$\underline{U}_h^k := \left\{ \underline{v}_h = ((v_T)_{T \in \mathcal{T}_h}, (v_F)_{F \in \mathcal{F}_h}) : v_T \in \mathbb{P}^k(T) \forall T \in \mathcal{T}_h, \right. \\ \left. v_F \in \mathbb{P}^k(F) \forall F \in \mathcal{F}_h \right\}$$

Interpolador global

$$\underline{I}_h^k : W^{1,1}(\Omega) \rightarrow \underline{U}_h^k \text{ tal que } \underline{I}_h^k v := \left( (\pi_T^{0,k} v)_{T \in \mathcal{T}_h}, (\pi_F^{0,k} v)_{F \in \mathcal{F}_h} \right)$$

Seminorma global  $\forall \underline{v}_h \in \underline{U}_h^k, \|\underline{v}_h\|_{1,h} := \left( \sum_{T \in \mathcal{T}_h} \|v_T\|_{1,T}^2 \right)^{\frac{1}{2}}$

# Problema discreto

Espacio GLOBAL de incógnitas discretas

$$\underline{U}_h^k := \left\{ \underline{v}_h = ((v_T)_{T \in \mathcal{T}_h}, (v_F)_{F \in \mathcal{F}_h}) : v_T \in \mathbb{P}^k(T) \forall T \in \mathcal{T}_h, \right. \\ \left. v_F \in \mathbb{P}^k(F) \forall F \in \mathcal{F}_h \right\}$$

Interpolador global

$$\underline{I}_h^k : W^{1,1}(\Omega) \rightarrow \underline{U}_h^k \text{ tal que } \underline{I}_h^k v := \left( (\pi_T^{0,k} v)_{T \in \mathcal{T}_h}, (\pi_F^{0,k} v)_{F \in \mathcal{F}_h} \right)$$

Seminorma global  $\forall \underline{v}_h \in \underline{U}_h^k, \|\underline{v}_h\|_{1,h} := \left( \sum_{T \in \mathcal{T}_h} \|\underline{v}_T\|_{1,T}^2 \right)^{\frac{1}{2}}$

$$\underline{U}_{h,0}^k := \{ \underline{v}_h \in \underline{U}_h^k : v_F = 0 \forall F \in \mathcal{F}_h^b \}$$

## Formas bilineales GLOBALES

$$a_h : \underline{U}_h^k \times \underline{U}_h^k \rightarrow \mathbb{R}$$
$$(\underline{u}_h, \underline{v}_h) \mapsto \sum_{T \in \mathcal{T}_h} a_T(\underline{u}_T, \underline{v}_T)$$

$$s_h : \underline{U}_h^k \times \underline{U}_h^k \rightarrow \mathbb{R}$$
$$(\underline{u}_h, \underline{v}_h) \mapsto \sum_{T \in \mathcal{T}_h} s_T(\underline{u}_T, \underline{v}_T)$$

## Esquema HHO para la aproximación del Problema Modelo

Hallar  $\underline{u}_h \in \underline{U}_{h,0}^k$  tal que

$$a_h(\underline{u}_h, \underline{v}_h) = (f, v_h) \quad \forall \underline{v}_h \in \underline{U}_{h,0}^k$$

→ El Problema Discreto posee *única solución* y la única solución discreta  $\underline{u}_h \in \underline{U}_{h,0}^k$  satisface la siguiente *acotación a priori*:

$$\|\underline{u}_h\|_{a,h} \lesssim \|f\|.$$



# Algunos resultados de convergencia para $k = 0$

- Estimación del Error en la Norma de la Energía Discreta

$$\|\underline{u}_h - \underline{I}_h^0 u\|_{a,h} \lesssim h|u|_{H^2(\mathcal{T}_h)}$$

- Estimación del Error en la Norma de la Energía para la Reconstrucción de la solución discreta

$$\|\nabla_h(p_h^1 \underline{u}_h - u)\| + |\underline{u}_h|_{s,h} \lesssim h|u|_{H^2(\mathcal{T}_h)}$$

donde la seminorma  $|\underline{u}_h|_{s,h} := s_h(\underline{u}_h, \underline{u}_h)^{\frac{1}{2}}$ .

- Estimación del Error en la Norma  $L^2$

$$\|p_h^1 \underline{u}_h - u\| \lesssim h^2 \|f\|_{H^1(\mathcal{T}_h)}$$

# Tabla de Contenidos

- ① Problema de convección-difusión *singularmente perturbado*
- ② Hybrid High-Order Method
- ③ Exponential Fitting
- ④ Implementaciones numéricas

$$\begin{cases} -\operatorname{div}(\varepsilon \nabla u - \bar{\beta} u) = f & \text{en } \Omega, \\ u = 0 & \text{en } \Gamma. \end{cases} \quad (1)$$

$$\begin{cases} -\operatorname{div}(\varepsilon \nabla u - \bar{\beta} u) = f & \text{en } \Omega, \\ u = 0 & \text{en } \Gamma. \end{cases} \quad (1)$$

Caso  $\bar{\beta} = \nabla \psi$  con  $\psi$  continua

$$\begin{cases} -\operatorname{div}(\varepsilon \nabla u - \bar{\beta} u) = f & \text{en } \Omega, \\ u = 0 & \text{en } \Gamma. \end{cases} \quad (1)$$

Caso  $\bar{\beta} = \nabla \psi$  con  $\psi$  continua

Simetrizamos (1), introduciendo la variable

$$\rho = u \exp\left(-\frac{\psi}{\varepsilon}\right) \rightarrow u = \rho \exp\left(\frac{\psi}{\varepsilon}\right)$$

$$\begin{cases} -\operatorname{div}(\varepsilon \nabla u - \bar{\beta} u) = f & \text{en } \Omega, \\ u = 0 & \text{en } \Gamma. \end{cases} \quad (1)$$

Caso  $\bar{\beta} = \nabla \psi$  con  $\psi$  continua

Simetrizamos (1), introduciendo la variable

$$\rho = u \exp\left(-\frac{\psi}{\varepsilon}\right) \rightarrow u = \rho \exp\left(\frac{\psi}{\varepsilon}\right)$$

y resulta

$$\begin{cases} -\operatorname{div}\left(\varepsilon \exp\left(\frac{\psi}{\varepsilon}\right) \nabla \rho\right) = f & \text{en } \Omega, \\ \rho = 0 & \text{en } \Gamma. \end{cases}$$

Aproximamos  $a := \varepsilon \exp\left(\frac{\psi}{\varepsilon}\right)$  por una constante en cada  $T$

Aproximamos  $a := \varepsilon \exp\left(\frac{\psi}{\varepsilon}\right)$  por una constante en cada  $T$   
→ Media armónica:

$$m_T = \frac{1}{\frac{1}{|T|} \iint_T \frac{1}{a} dA} = \frac{|T|\varepsilon}{\iint_T \exp\left(-\frac{\psi}{\varepsilon}\right) dA}$$



Aproximamos  $a := \varepsilon \exp\left(\frac{\psi}{\varepsilon}\right)$  por una constante en cada  $T$   
 → Media armónica:

$$m_T = \frac{1}{\frac{1}{|T|} \iint_T \frac{1}{a} dA} = \frac{|T|\varepsilon}{\iint_T \exp\left(-\frac{\psi}{\varepsilon}\right) dA}$$

Obtenemos el problema

$$\begin{cases} -\operatorname{div}(M\nabla\rho) = f & \text{en } \Omega, \\ \rho = 0 & \text{en } \Gamma, \end{cases}$$

donde  $M|_T = m_T \forall T \in \mathcal{T}_h$ .

Hallar  $\rho_h \in \underline{U}_{h,0}^0$  tal que

$$a_h(\underline{\rho}_h, \underline{v}_h) = (f, v_h) \quad \forall \underline{v}_h \in \underline{U}_{h,0}^0$$

Hallar  $\rho_h \in \underline{U}_{h,0}^0$  tal que

$$a_h(\underline{\rho}_h, \underline{v}_h) = (f, v_h) \quad \forall \underline{v}_h \in \underline{U}_{h,0}^0$$

Consideramos una inversa discreta  $\mathcal{R}$  de  $u \rightarrow \rho = u \exp\left(\frac{\psi}{\varepsilon}\right)$  definida por

$$\mathcal{R} : \underline{\rho}_h \rightarrow \underline{u}_h \quad \text{s. t.} \quad \underline{\rho}_h = ((\rho_T)_{T \in \mathcal{T}_h}, (\rho_F)_{F \in \mathcal{F}_h}) \quad \text{donde}$$

$$\rho_T = \left( \frac{1}{|\mathcal{F}_T|} \sum_{F \in \mathcal{F}_T} \int_F e^{-\frac{\psi}{\varepsilon}|_T} ds \right) u_T, \quad \rho_F = \left( \int_F e^{-\frac{\psi}{\varepsilon}} ds \right) u_F$$

Hallar  $\rho_h \in \underline{U}_{h,0}^0$  tal que

$$a_h(\underline{\rho}_h, \underline{v}_h) = (f, v_h) \quad \forall \underline{v}_h \in \underline{U}_{h,0}^0$$

Consideramos una inversa discreta  $\mathcal{R}$  de  $u \rightarrow \rho = u \exp\left(\frac{\psi}{\varepsilon}\right)$  definida por

$$\mathcal{R} : \underline{\rho}_h \rightarrow \underline{u}_h \quad \text{s. t.} \quad \underline{\rho}_h = ((\rho_T)_{T \in \mathcal{T}_h}, (\rho_F)_{F \in \mathcal{F}_h}) \quad \text{donde}$$

$$\rho_T = \left( \frac{1}{|\mathcal{F}_T|} \sum_{F \in \mathcal{F}_T} \int_F e^{-\frac{\psi}{\varepsilon}|_T} ds \right) u_T, \quad \rho_F = \left( \int_F e^{-\frac{\psi}{\varepsilon}} ds \right) u_F$$

Hallar  $\mathcal{R}\underline{u}_h \in \underline{U}_{h,0}^0$  tal que

$$a_h(\mathcal{R}\underline{u}_h, \underline{v}_h) = (f, v_h) \quad \forall \underline{v}_h \in \underline{U}_{h,0}^0$$

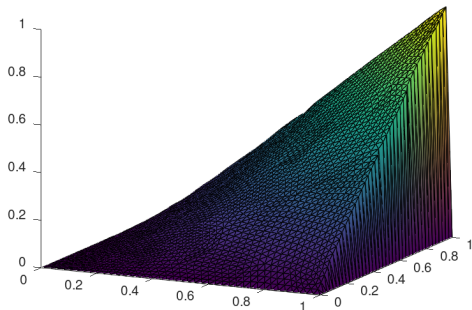
# Tabla de Contenidos

- 1 Problema de convección-difusión *singularmente perturbado*
- 2 Hybrid High-Order Method
- 3 Exponential Fitting
- 4 Implementaciones numéricas

# Ejemplo 1

$$\epsilon = 10^{-6}, \Omega = (0, 1) \times (0, 1), \bar{\beta} = [1, 1] \rightarrow \psi(x, y) = x + y$$

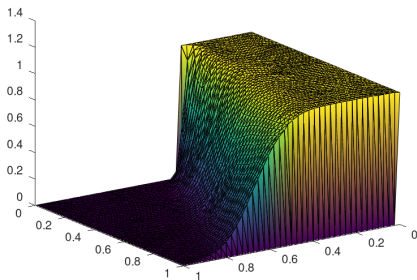
$$f(x, y) = x - \frac{e^{\frac{x-1}{\epsilon}} - e^{-\frac{1}{\epsilon}}}{1 - e^{-\frac{1}{\epsilon}}} + y - \frac{e^{\frac{y-1}{\epsilon}} - e^{-\frac{1}{\epsilon}}}{1 - e^{-\frac{1}{\epsilon}}}$$



Condiciones Dirichlet homogéneas.

## Ejemplo 2

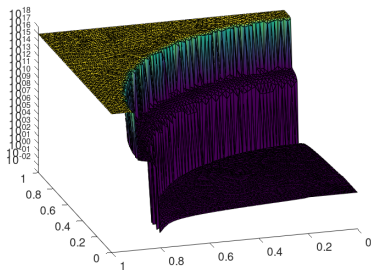
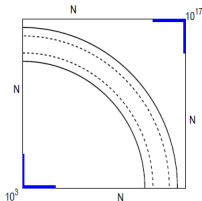
$$\epsilon = 10^{-6}, \quad \Omega = (0, 1) \times (0, 1), \quad \bar{\beta} = \left[\frac{1}{3}, 1\right]$$



$$f(x, y) = 0, \quad g(x, y) = \begin{cases} 1 & \text{si } x = 0 \text{ ó } x < \frac{1}{3}, y = 0 \\ 0 & \text{si no.} \end{cases}$$

# Ejemplo 3

$$\epsilon = 10^{-4}, \Omega = (0, 1) \times (0, 1), \bar{\beta} = \nabla \Psi$$



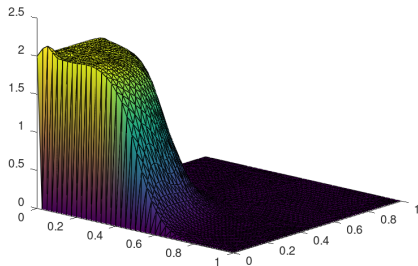
$$\psi(x, y) = \begin{cases} 0 & \text{si } 0 \leq \sqrt{x^2 + y^2} \leq 0.8 \\ 2(\sqrt{x^2 + y^2} - 0.8) & \text{si } 0.8 \leq \sqrt{x^2 + y^2} \leq 0.9 \\ 0.2 & \text{si } \sqrt{x^2 + y^2} \geq 0.9, \end{cases} \quad f(x, y) = \begin{cases} 0.5 \cdot 10^{12}, & \text{si } 0.82 \leq \sqrt{x^2 + y^2} \leq 0.88 \\ 0 & \text{si no.} \end{cases}$$

Condiciones de Neumann nulas



## Ejemplo 4

$$\epsilon = 10^{-6}, \quad \Omega = (0, 1) \times (0, 1), \quad \bar{\beta}(x, y) = (2y(1 - x^2), -2x(1 - y^2))$$



$$f(x, y) = 0, \quad g(x, y) = \begin{cases} 1 + \tanh(10(-2y + 1)) & \text{si } x = 0 \\ 0 & \text{si no.} \end{cases}$$

En cada elemento aproximamos  $\bar{\beta} \sim \nabla\psi$ , donde

$\psi(x, y) = 2y_K(1 - x_K^2)x - 2x_K(1 - y_K^2)y$ , siendo  $(x_K, y_K)$  el baricentro de  $K$ .

- [1] BREZZI, F., MARINI, L. D., AND PIETRA, P. Two-dimensional exponential fitting and applications to drift-diffusion models. *SIAM Journal on Numerical Analysis* 26, 6 (1989), 1342–1355.
- [2] DI PIETRO, D. A., AND DRONIOU, J. The hybrid high-order method for polytopal meshes. *Design, analysis, and applications* 19 (2019).
- [3] DI PIETRO, D. A., DRONIOU, J., AND ERN, A. A discontinuous-skeletal method for advection-diffusion-reaction on general meshes. *SIAM Journal on Numerical Analysis* 53, 5 (2015), 2135–2157.
- [4] LOMBARDI, A. L., AND PIETRA, P. Exponentially fitted discontinuous galerkin schemes for singularly perturbed problems. *Numerical Methods for Partial Differential Equations* 28, 6 (2012), 1747–1777.

# ¡Gracias!