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Un problema de Stefan a dos fases no clásico con coeficientes térmicos variables

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Esquema

1. Planteo del problema
2. Formulación integral
3. Existencia de solución
4. Casos particulares

Planteo del problema

Problema (P)

$$\rho_1(T)c_1(T)\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(k_1(T)\tfrac{\partial T}{\partial x}\right) - v_1(T)\frac{\partial T}{\partial x} - F_1(T), \quad 0 < x < s(t), \quad t > 0,$$

$$\rho_2(U)c_2(U)\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\left(k_2(U)\tfrac{\partial U}{\partial x}\right) - v_2(U)\frac{\partial U}{\partial x} - F_2(U), \quad x > s(t), \quad t > 0,$$

$$T(0,t) = T^*, \qquad t > 0,$$

$$T(s(t),t) = U(s(t),t) = T_m, \qquad t > 0,$$

$$U(x,0) = U(+\infty,t) = U_0, \qquad t > 0,$$

$$k_2(T_m)\frac{\partial U}{\partial x}(s(t),t) - k_1(T_m)\frac{\partial T}{\partial x}(s(t),t) = \rho_0 \ell \dot{s}(t), \qquad t > 0,$$

$$s(0) = 0,$$

Hipótesis

$$U_0 < T_m < T^*,$$

$$v_1(T) = \frac{\mu_1(T)}{\sqrt{t}}, \qquad v_2(U) = \frac{\mu_2(U)}{\sqrt{t}},$$

$$F_1(T) = \frac{\beta_1(T)}{t}, \qquad F_2(U) = \frac{\beta_2(U)}{t}.$$

Valores de referencia, $i = 1, 2$

k_{0i} : conductividad térmica

ρ_0 : densidad de masa

c_{0i} : calor específico

$\alpha_{0i} := \dfrac{k_{0i}}{\rho_0 c_{0i}}$: difusividad térmica.

Variable de similaridad

$$\xi = \frac{x}{2\sqrt{\alpha_{01} t}}$$

Transformaciones

$$f(\xi) = \frac{T(x,t) - T_m}{T^* - T_m}, \qquad g(\xi) = \frac{U(x,t) - T_m}{T_m - U_0}$$

Frontera Libre

$$s(t) = 2\lambda\sqrt{\alpha_{01} t}$$

Problema (PDO)

$$\Big(L_1^*(f)f'\Big)' + 2f'\Big(N_1^*(f)\,\xi - \mu_1^*(f)\Big) - \beta_1^*(f) = 0, \qquad 0 < \xi < \lambda,$$

$$\frac{\alpha_{02}}{\alpha_{01}}\Big(L_2^*(g)g'\Big)' + 2g'\Big(N_2^*(g)\,\xi - \mu_2^*(g)\Big) - \beta_2^*(g) = 0, \qquad \xi > \lambda,$$

$$f(0) = 1,$$

$$f(\lambda) = g(\lambda) = 0,$$

$$g(+\infty) = -1,$$

$$\alpha_{02}\ \mathrm{Ste}_2\ L_2^*(g)(\lambda)g'(\lambda) - \alpha_{01}\ \mathrm{Ste}_1\ L_1^*(f)(\lambda)f'(\lambda) = 2\lambda\alpha_{01},$$

Datos

$$L_1^*(f) = \frac{k_1(T)}{k_{01}} \qquad N_1^*(f) = \frac{\rho_1(T)c_1(T)}{\rho_0 c_{01}} \qquad \mu_1^*(f) = \frac{\mu_1(T)}{\sqrt{\alpha_{01}\rho_0 c_{01}}} \qquad \beta_1^*(f) = \frac{4\beta_1(T)}{(T^* - T_m)\rho_0 c_{01}}$$

$$L_2^*(g) = \frac{k_2(U)}{k_{02}} \qquad N_2^*(g) = \frac{\rho_2(U)c_2(U)}{\rho_0 c_{02}} \qquad \mu_2^*(g) = \frac{\mu_2(U)}{\sqrt{\alpha_{01}\rho_0 c_{02}}} \qquad \beta_2^*(g) = \frac{4\beta_2(U)}{(T_m - U_0)\rho_0 c_{02}}.$$

$$\mathrm{Ste}_1 = \frac{c_{01}(T^* - T_m)}{\ell} \qquad \mathrm{Ste}_2 = \frac{c_{02}(T_m - U_0)}{\ell}$$

Problema equivalente

Sistema de ecuaciones integrales

$$f(\xi) = 1 + \chi_1(f)(\xi) - \frac{\Phi_1(f)(\xi)}{\Phi_1(f)(\lambda)} \left(1 + \chi_1(f)(\lambda)\right), \qquad 0 \leq \xi \leq \lambda$$

$$g(\xi) = \chi_2(g)(\xi) - \frac{\Phi_2(g)(\xi)}{\Phi_2(g)(+\infty)} \left(1 + \chi_2(g)(+\infty)\right), \qquad \xi \geq \lambda$$

acoplado a la siguiente condición

$$-\mathrm{Ste}_2 \, \frac{1 + \chi_2(g)(\infty)}{\phi_2(g)(\infty)} + \mathrm{Ste}_1 E_1(f)(\lambda) \left(\frac{1 + \chi_1(f)(\lambda)}{\phi_1(f)(\lambda)} - w_1(f)(\lambda) \right) = 2\lambda.$$

$$\Phi_1(f)(\xi) = \int_0^{\xi} \frac{E_1(f)(z)}{L_1^*(f)(z)}\,\mathrm{d}z, \qquad \Phi_2(g)(\xi) = \frac{\alpha_{01}}{\alpha_{02}} \int_{\lambda}^{\xi} \frac{E_2(g)(z)}{L_2^*(g)(z)}\,\mathrm{d}z$$

$$E_1(f)(\xi) = \frac{U_1(f)(\xi)}{I_1(f)(\xi)}, \qquad E_2(g)(\xi) = \frac{U_2(g)(\xi)}{I_2(g)(\xi)},$$

$$U_1(f)(\xi) = \exp\Big(2\int_0^{\xi} \frac{\mu_1^*(f)(z)}{L_1^*(f)(z)}\,\mathrm{d}z\Big), \qquad U_2(g)(\xi) = \exp\Big(2\frac{\alpha_{01}}{\alpha_{02}}\int_{\lambda}^{\xi} \frac{\mu_2^*(g)(z)}{L_2^*(g)(z)}\,\mathrm{d}z\Big),$$

$$I_1(f)(\xi) = \exp\Big(2\int_0^{\xi} \frac{zN_1^*(f)(z)}{L_1^*(f)(z)}\,\mathrm{d}z\Big), \qquad I_2(g)(\xi) = \exp\Big(2\frac{\alpha_{01}}{\alpha_{02}}\int_{\lambda}^{\xi} \frac{zN_2^*(g)(z)}{L_2^*(g)(z)}\,\mathrm{d}z\Big),$$

$$\chi_1(f)(\xi) = \int_0^{\xi} \frac{E_1(f)(z)w_1(f)(z)}{L_1^*(f)(z)}\,\mathrm{d}z, \qquad \chi_2(g)(\xi) = \frac{\alpha_{01}}{\alpha_{02}}\int_{\lambda}^{\xi} \frac{E_2(g)(z)w_2(g)(z)}{L_2^*(g)(z)}\,\mathrm{d}z,$$

$$w_1(f)(\xi) = \int_0^{\xi} \frac{\beta_1^*(f)(z)}{E_1(f)(z)}\,\mathrm{d}z, \qquad w_2(g)(\xi) = \int_{\lambda}^{\xi} \frac{\beta_2^*(g)(z)}{E_2(g)(z)}\,\mathrm{d}z.$$

Sea $\lambda > 0$ fijo.

Operadores

- $\forall f \in \mathcal{F} := C^0[0,\lambda]$, se define:

$$\mathcal{H}_1(f)(\xi) = 1 + \chi_1(f)(\xi) - \frac{\Phi_1(f)(\xi)}{\Phi_1(f)(\lambda)}\left(1 + \chi_1(f)(\lambda)\right), \qquad 0 \leq \xi \leq \lambda.$$

- $\forall g \in \mathcal{G} := \{g \in C^0[\lambda,+\infty) : g(\lambda) = 0,\ g(+\infty) = -1\}$ se define:

$$\mathcal{H}_2(g)(\xi) = \chi_2(g)(\xi) - \frac{\Phi_2(g)(\xi)}{\Phi_2(g)(+\infty)}\left(1 + \chi_2(g)(+\infty)\right), \qquad \xi \geq \lambda.$$

Observación

- $\mathcal{F}$ es un espacio de Banach dotado con la norma del máximo $\|f\| = \max\limits_{\xi\in[0,\lambda]} |f(\xi)|$.
- $\mathcal{G}$ es un subconjunto cerrado del espacio de Banach de las funciones continuas y acotadas en $[\lambda,\infty)$ con la norma del supremo.

Problema a resolver (PI)

$$\mathcal{H}_1(f)(\xi) = f(\xi), \qquad 0 \leq \xi \leq \lambda \tag{1}$$

$$\mathcal{H}_2(g)(\xi) = g(\xi), \qquad \xi \geq \lambda \tag{2}$$

$$-\,\mathrm{Ste}_2\,\frac{1+\chi_2(g)(\infty)}{\phi_2(g)(\infty)} + \mathrm{Ste}_1 E_1(f)(\lambda)\left(\frac{1+\chi_1(f)(\lambda)}{\phi_1(f)(\lambda)} - w_1(f)(\lambda)\right) = 2\lambda. \tag{3}$$

$$\begin{cases} \text{Existen } L_{im} > 0 \text{ , } L_{iM} > 0 \text{ y } \widetilde{L}_i > 0 \text{ tales que} \\ L_{im} \leq L_i^*(h(\xi)) \leq L_{iM}, \qquad \forall h \in C^0(\mathbb{R}_0^+) \cap L^\infty(\mathbb{R}_0^+),\ \forall \xi \in \mathbb{R}_0^+. \\ |L_i^*(h_1(\xi)) - L_i^*(h_2(\xi))| \leq \widetilde{L}_i \|h_1 - h_2\|,\ \ \forall h_1, h_2 \in C^0(\mathbb{R}_0^+) \cap L^\infty(\mathbb{R}_0^+) \quad \forall \xi \in \mathbb{R}_0^+. \end{cases}$$

$$\begin{cases} \text{Existen } N_{im} > 0 \text{ , } N_{iM} > 0 \text{ y } \widetilde{N}_i > 0 \text{ tales que} \\ N_{im} \leq N_i^*(h(\xi)) \leq N_{iM}, \quad \forall h \in C^0(\mathbb{R}_0^+) \cap L^\infty(\mathbb{R}_0^+),\ \xi \in \mathbb{R}_0^+. \\ |N_i^*(h_1(\xi)) - N_i^*(h_2(\xi))| \leq \widetilde{N}_i \|h_1 - h_2\|,\ \ \forall h_1, h_2 \in C^0(\mathbb{R}_0^+) \cap L^\infty(\mathbb{R}_0^+),\ \xi \in \mathbb{R}_0^+. \end{cases}$$

$$\begin{cases} \text{Existen } \mu_{im} > 0 \text{ , } \mu_{iM} > 0 \text{ y } \widetilde{\mu}_i > 0 \text{ tales que} \\ \mu_{im} \leq \mu_i^*(h(\xi)) \leq \mu_{iM}, \qquad \forall h \in C^0(\mathbb{R}_0^+) \cap L^\infty(\mathbb{R}_0^+),\ \xi \in \mathbb{R}_0^+. \\ |\mu_i^*(h_1(\xi)) - \mu_i^*(h_2(\xi))| \leq \widetilde{\mu}_i \ \|h_1 - h_2\|,\ \ \forall h_1, h_2 \in C^0(\mathbb{R}_0^+) \cap L^\infty(\mathbb{R}_0^+),\ \xi \in \mathbb{R}_0^+. \end{cases}$$

$$\begin{cases} \text{Existen } \beta_{1m} > 0 \text{ , } \beta_{1M} > 0 \text{ y } \widetilde{\beta}_1 > 0 \text{ tales que} \\ \beta_{1m} \leq \beta_1^*(h(\xi)) \leq \beta_{1M}, \qquad \forall h \in C^0(\mathbb{R}_0^+) \cap L^\infty(\mathbb{R}_0^+),\ \xi \in \mathbb{R}_0^+. \\ |\beta_1^*(h_1(\xi)) - \beta_1^*(h_2(\xi))| \leq \widetilde{\beta}_1 \ \|h_1 - h_2\|,\ \ \forall h_1, h_2 \in C^0(\mathbb{R}_0^+) \cap L^\infty(\mathbb{R}_0^+),\ \xi \in \mathbb{R}_0^+. \end{cases}$$

$$\begin{cases} \text{Existen } R > P \text{ con } P = \frac{\alpha_{01} N_{2M}}{\alpha_{02} L_{2m}} \text{ y } \widetilde{\beta}_2(\xi) \leq \exp(-R\ \xi^2), \quad \forall\ \xi \in \mathbb{R}_0^+. \text{ tales que} \\ \beta_2^*(h(\xi)) \leq \exp(-R\ \xi^2), \qquad \forall h \in C^0(\mathbb{R}_0^+) \cap L^\infty(\mathbb{R}_0^+),\ \xi \in \mathbb{R}_0^+. \\ |\beta_2^*(h_1(\xi)) - \beta_2^*(h_2(\xi))| \leq \widetilde{\beta}_2(\xi)\ \|h_1 - h_2\|,\ \ \forall h_1, h_2 \in C^0(\mathbb{R}_0^+) \cap L^\infty(\mathbb{R}_0^+),\ \xi \in \mathbb{R}_0^+. \end{cases}$$

Lema

Para todo $\xi \in [0,\lambda]$, $f \in \mathcal{F} = C^0[0,\lambda]$, se tienen las siguientes desigualdades:

$$\exp\left(-\frac{N_{1M}}{L_{1m}}\xi^2\right) \leq \frac{\exp\left(2\frac{\mu_{1m}}{L_{1M}}\xi\right)}{\exp\left(\frac{N_{1M}}{L_{1m}}\xi^2\right)} \leq E_1(f)(\xi) \leq \frac{\exp\left(2\frac{\mu_{1M}}{L_{1m}}\xi\right)}{\exp\left(\frac{N_{1m}}{L_{1M}}\xi^2\right)} \leq \exp\left(2\frac{\mu_{1M}}{L_{1m}}\xi\right),$$

$$\Phi_1(f)(\xi) \leq \frac{\sqrt{L_{1M}}\exp\left(\frac{\mu_{1M}^2 L_{1M}}{L_{1m}^2 N_{1m}}\right)\left[\operatorname{erf}\left(\sqrt{\frac{N_{1m}}{L_{1M}}}\xi - \sqrt{\frac{L_{1M}}{N_{1m}}}\frac{\mu_{1M}}{L_{1m}}\right)\right) + \operatorname{erf}\left(\sqrt{\frac{L_{1M}}{N_{1m}}}\frac{\mu_{1M}}{L_{1m}}\right)\right]}{L_{1m}\sqrt{N_{1m}}}$$

$$\leq \frac{\exp\left(2\frac{\mu_{1M}}{L_{1m}}\xi\right)}{2\mu_{1M}},$$

$$\Phi_1(f)(\xi) \geq \frac{\sqrt{\pi}}{2}\frac{\sqrt{L_{1m}}}{L_{1M}\sqrt{N_{1M}}}\operatorname{erf}\left(\sqrt{\frac{N_{1M}}{L_{1m}}}\xi\right) \geq \frac{\xi}{L_{1M}}\exp\left(-\frac{N_{1M}}{L_{1m}}\xi^2\right),$$

$$\frac{\beta_{1m}L_{1m}}{2\mu_{1M}}\left(1-\exp\left(-\frac{2\mu_{1M}}{L_{1m}}\xi\right)\right) \leq w_1(f)(\xi) \leq \beta_{1M}\xi\exp\left(\frac{N_{1M}}{L_{1m}}\xi^2\right),$$

$$\chi_1(f)(\xi) \leq \exp\left(\frac{2\mu_{1M}}{L_{1m}}\xi + \frac{N_{1M}}{L_{1m}}\xi^2\right)\frac{\xi^2\beta_{1M}}{L_{1m}},$$

$$\chi_1(f)(\xi) \geq \frac{\beta_{1m}L_{1m}}{2\mu_{1M}L_{1M}}\left[\frac{\sqrt{\pi}}{2}\sqrt{\frac{L_{1m}}{N_{1M}}}\operatorname{erf}\left(\sqrt{\frac{N_{1M}}{L_{1m}}}\xi\right) - \xi\exp\left(-\frac{2\mu_{1M}\xi + N_{1M}\xi^2}{L_{1m}}\right)\right].$$

Lema

Para cada $\xi \in [\lambda, +\infty]$, $g \in \mathcal{G}$, *tenemos*

$$\exp\left(-\frac{\alpha_{01}}{\alpha_{02}}\frac{N_{2M}}{L_{2m}}(\xi^2-\lambda^2)\right) \leq E_2(g)(\xi) \leq \exp\left(\frac{\alpha_{01}}{\alpha_{02}}\left[\frac{2\mu_{2M}(\xi-\lambda)-N_{2m}(\xi^2-\lambda^2)}{L_{2M}}\right]\right),$$

$$\Phi_2(g)(\xi) \leq \frac{\sqrt{\pi}}{2}\sqrt{\frac{\alpha_{01}}{\alpha_{02}}\frac{L_{2M}}{N_{2m}}}\exp\left(\frac{\alpha_{01}}{\alpha_{02}}\frac{N_{2m}}{L_{2M}}\left(\frac{\lambda-\mu_{2M}}{N_{2m}}\right)^2\right)\left[\operatorname{erf}\left(\sqrt{\frac{\alpha_{01}}{\alpha_{02}}\frac{N_{2m}}{L_{2M}}}\left(\xi-\frac{\mu_{2M}}{N_{2m}}\right)\right)\right.$$
$$\left.-\operatorname{erf}\left(\sqrt{\frac{\alpha_{01}}{\alpha_{02}}\frac{N_{2m}}{L_{2M}}}\left(\lambda-\frac{\mu_{2M}}{N_{2m}}\right)\right)\right],$$

$$\Phi_2(g)(\xi) \geq \frac{\sqrt{\pi}}{2}\sqrt{\frac{\alpha_{01}}{\alpha_{02}}\frac{L_{2m}}{N_{2M}}}\exp\left(\frac{\alpha_{01}}{\alpha_{02}}\frac{N_{2M}}{L_{2m}}\lambda^2\right)\left[\operatorname{erf}\left(\sqrt{\frac{\alpha_{01}}{\alpha_{02}}\frac{N_{2M}}{L_{2m}}}\xi\right)-\operatorname{erf}\left(\sqrt{\frac{\alpha_{01}}{\alpha_{02}}\frac{N_{2M}}{L_{2m}}}\lambda\right)\right],$$

$$\chi_2(g)(\xi) \leq \frac{\alpha_{01}}{\alpha_{02}}\frac{\overline{A}}{2L_{2m}(R-P)}\left(\exp(-(R-P)\lambda^2)-\exp(-(R-P)\xi^2)\right),$$

$$w_2(g)(\xi) \leq \frac{\sqrt{\pi}}{2}\frac{\exp(-P\lambda^2)}{\sqrt{R-P}}\operatorname{erfc}(\sqrt{R-P}\lambda),$$

donde

$$\overline{A} = \exp\left(\frac{\alpha_{01}}{\alpha_{02}}\left(\frac{\mu_{2M}^2}{L_{2M}L_{2m}}+\frac{\mu_{2m}^2}{N_{2m}L_{2M}}\right)\right). \tag{4}$$

Lema

Sea $\lambda > 0$*, para cada* $\xi \in [0,\lambda]$ *y* $f_1, f_2 \in C^0[0,\lambda]$ *se tiene:*

$$|U_1(f_1)(\xi) - U_1(f_2)(\xi)| \leq D_{11}(\lambda)\|f_1 - f_2\|,$$

$$|I_1(f_1)(\xi) - I_2(f_2)(\xi)| \leq D_{12}(\lambda)\|f_1 - f_2\|,$$

$$|E_1(f_1)(\xi) - E_1(f_2)(\xi)| \leq D_{13}(\lambda)\|f_1 - f_2\|,$$

$$|\Phi_1(f_1)(\xi) - \Phi_2(f_2)(\xi)| \leq \lambda\, D_{14}(\lambda)\|f_1 - f_2\|,$$

$$|w_1(f_1)(\xi) - w_1(f_2)(\xi)| \leq D_{15}(\lambda)\|f_1 - f_2\|,$$

$$|\chi_1(f_1)(\xi) - \chi_1(f_2)(\xi)| \leq \lambda\, D_{16}(\lambda)\|f_1 - f_2\|,$$

$$D_{11}(\lambda) = \frac{2\exp\left(\frac{2\mu_{1M}}{L_{1m}}\right)}{L_{1m}^2}\lambda\left(\mu_{1M}\widetilde{L}_1 + L_{1m}\widetilde{\mu}_1\right), \qquad D_{12}(\lambda) = \frac{\exp\left(\frac{N_{1M}}{L_{1m}}\lambda^2\right)}{L_{1m}^2}\lambda^2\left(N_{1M}\widetilde{L}_1 + L_{1m}\widetilde{N}_1\right),$$

$$D_{13}(\lambda) = \exp\left(\frac{N_{1M}}{L_{1m}}\lambda^2\right)D_{11}(\lambda) + \exp\left(2\frac{\mu_{1M}}{L_{1m}}\lambda\right)D_{12}(\lambda), \qquad D_{14}(\lambda) = \frac{1}{L_{1m}^2}\left(\widetilde{L}_1\exp\left(2\lambda\frac{\mu_{1M}}{L_{1m}}\right) + L_{1m}D_{13}(\lambda)\right),$$

$$D_{15}(\lambda) = \lambda\exp\left(\frac{N_{1M}}{L_{1m}}\lambda^2\right)\left(\widetilde{\beta}_1 + \beta_{1M}\exp\left(\frac{N_{1M}}{L_{1m}}\lambda^2\right)D_{13}(\lambda)\right),$$

$$D_{16}(\lambda) = \frac{\exp\left(2\frac{\mu_{1M}}{L_{1m}}\lambda\right)}{L_{1m}}\lambda D_{15}(\lambda) + \frac{\beta_{1M}}{L_{1m}}\lambda^2\exp\left(\frac{N_{1M}}{L_{1m}}\lambda^2\right)\left(D_{13}(\lambda) + \frac{\widetilde{L}_1}{L_{1m}}\exp\left(2\frac{\mu_{1M}}{L_{1m}}\lambda\right)\right).$$

Lema

Dado $\lambda > 0$, para cada $\xi \in [\lambda, +\infty]$ y $g_1, g_2 \in C_b(\lambda)$ se tiene

$$|E_2(g_1)(\xi) - E_2(g_2)(\xi)| \leq D_{21}(\xi, \lambda)\|g_1 - g_2\|,$$

$$|\Phi_2(g_1)(\xi) - \Phi_2(g_2)(\xi)| \leq D_{22}(\lambda)\|g_1 - g_2\|,$$

$$|w_2(g_1)(\xi) - w_2(g_2)(\xi)| \leq D_{23}(\lambda)\|g_1 - g_2\|,$$

$$|\chi_2(g_1)(\xi) - \chi_2(g_2)(\xi)| \leq D_{24}(\lambda)\|g_1 - g_2\|,$$

$$D_{21}(\xi, \lambda) = \frac{\alpha_{01}}{\alpha_{02}} \left[2a(\xi - \lambda) + b(\xi^2 - \lambda^2)\right] \exp\left(2\frac{\alpha_{01}}{\alpha_{02}} \left[\frac{\mu_{2M}}{L_{2m}}(\xi - \lambda) - \frac{N_{2m}}{L_{2m}} \frac{(\xi^2 - \lambda^2)}{2}\right]\right),$$

$$D_{22}(\lambda) = c \exp\left(\frac{\alpha_{01}}{\alpha_{02}} \frac{N_{2m}}{L_{2M}} \lambda^2\right) + \widetilde{c}(\lambda) \left[1 + \frac{\sqrt{\pi}\beta a}{\sqrt{\alpha}} + \frac{b}{4\alpha^{3/2}} \left[\sqrt{\pi}(1 + 2\alpha\beta^2) + 2\sqrt{\alpha}(\lambda + \beta)\right]\right],$$

$$D_{23}(\lambda) = \widetilde{d}(\lambda) \left[\frac{2a}{p} + \frac{aq}{p^{3/2}} \frac{\sqrt{\pi}}{2} + b\left(\frac{q}{2p^2} + \frac{\lambda}{2p} + \frac{\sqrt{\pi}}{2\sqrt{p}}\left(\frac{q^2}{4p^2} + \frac{1}{2p}\right)\right)\right],$$

$$D_{24}(\lambda) = \frac{\sqrt{\pi}}{2\sqrt{R-P}} D_{22}(\lambda) + D_{23}(\lambda) \frac{\sqrt{\pi}}{2} \sqrt{\frac{\alpha_{01}}{\alpha_{02}} \frac{L_{2M}}{N_{2m}}} \exp\left(\frac{\alpha_{01}}{\alpha_{02}} \frac{\lambda^2}{L_{2M} N_{2m}}\right),$$

Lema

El operador $\mathcal{H}_1 : \mathcal{F} \to \mathcal{F}$ *satisface* $\forall\ f_1, f_2 \in \mathcal{F}$, *la siguiente desigualdad*

$$||\mathcal{H}_1(f_1) - \mathcal{H}_1(f_2)|| \leq \mathcal{E}_1(\lambda)||f_1 - f_2||,$$

donde

$$\mathcal{E}_1(\lambda) = 2\lambda D_{16}(\lambda) + D^*(\lambda)\left(1 + D^\dagger(\lambda)\right)$$

con $D^*(\lambda) = 2L_{1M} \exp\left(\frac{N_{1M}}{L_{1m}}\lambda^2\right) D_{14}(\lambda),\ D^\dagger(\lambda) = \exp\left(\frac{2\mu_{1M}}{L_{1m}}\lambda + \frac{N_{1M}}{L_{1m}}\lambda^2\right)\frac{\lambda^2\beta_{1M}}{L_{1m}}$. *Además,* $\mathcal{E}_1$ *verifica*

$$\mathcal{E}_1(0) = \frac{2L_{1M}\widetilde{L}_1}{L_{1m}^2}, \qquad \mathcal{E}_1(+\infty) = +\infty, \qquad \mathcal{E}_1'(\lambda) > 0, \quad \forall\ \lambda > 0.$$

Teorema

Asumiendo (A): $\frac{2L_{1M}\widetilde{L}_1}{L_{1m}^2} < 1$ *existe un único* $\overline{\lambda}_1 = \mathcal{E}_1^{-1}(1)$ *tal que para cada* $0 < \lambda < \overline{\lambda}_1$, *el operador* $\mathcal{H}_1$ *tiene un único punto fijo* $f_\lambda \in \mathcal{F}$.

Esquema de la demostración del Lema

$$
\begin{aligned}
|\mathcal{H}_1(f_1)(\xi) - \mathcal{H}_1(f_2)(\xi)| &\leq |\chi_1(f_1)(\xi) - \chi_1(f_2)(\xi)| + \left|\tfrac{\Phi_1(f_1)(\xi)}{\Phi_1(f_1)(\lambda)} - \tfrac{\Phi_1(f_2)(\xi)}{\Phi_1(f_2)(\lambda)}\right| \\
&+ \left|\chi_1(f_1)(\xi)\frac{\Phi_1(f_1)(\xi)}{\Phi_1(f_1)(\lambda)} - \chi_1(f_2)(\xi)\frac{\Phi_1(f_2)(\xi)}{\Phi_1(f_2)(\lambda)}\right|.
\end{aligned}
$$

Se tiene

$$
\begin{aligned}
\left|\tfrac{\Phi_1(f_1)(\xi)}{\Phi_1(f_1)(\lambda)} - \tfrac{\Phi_1(f_2)(\xi)}{\Phi_1(f_2)(\lambda)}\right| &\leq \left|\tfrac{\Phi_1(f_1)(\xi)-\Phi_1(f_2)(\xi)}{\Phi_1(f_1)(\lambda)}\right| + \left|\tfrac{\Phi_1(f_2)(\xi)}{\Phi_1(f_2)(\lambda)}\right| \left|\tfrac{\Phi_1(f_1)(\lambda)-\Phi_1(f_2)(\lambda)}{\Phi_1(f_1)(\lambda)}\right| \\
&\leq 2\left|\tfrac{\Phi_1(f_1)(\lambda)-\Phi_1(f_2)(\lambda)}{\Phi_1(f_1)(\lambda)}\right| \leq D^*(\lambda)||f_1 - f_2||
\end{aligned}
$$

y

$$
\left|\chi_1(f_1)(\xi)\frac{\Phi_1(f_1)(\xi)}{\Phi_1(f_1)(\lambda)} - \chi_1(f_2)(\xi)\frac{\Phi_1(f_2)(\xi)}{\Phi_1(f_2)(\lambda)}\right| \leq \left|\frac{\Phi_1(f_1)(\xi)}{\Phi_1(f_1)(\lambda)}\right| |\chi_1(f_1)(\xi) - \chi_1(f_2)(\xi)|
$$

$$
+|\chi_1(f_2)(\xi)| \left|\tfrac{\Phi_1(f_1)(\xi)}{\Phi_1(f_1)(\lambda)} - \tfrac{\Phi_1(f_2)(\xi)}{\Phi_1(f_2)(\lambda)}\right| \leq \left(\lambda D_{16}(\lambda) + D^\dagger(\lambda)D^*(\lambda)\right)||f_1 - f_2||.
$$

Aplicando los lemas anteriores se obtiene la tesis.

□

Lema

El operador $\mathcal{H}_2 : \mathcal{G} \to \mathcal{G}$ *satisface* $\forall\, g_1, g_2 \in \mathcal{G}$, *la siguiente desigualdad*

$$||\mathcal{H}_2(g_1) - \mathcal{H}_2(g_2)|| \leq \mathcal{E}_2(\lambda)||g_1 - g_2||,$$

donde $\mathcal{E}_2(\lambda) = 2D_{24}(\lambda) + \left(1 + \frac{\alpha_{01}}{\alpha_{02}} \frac{\overline{A}}{2L_{2m}(R-P)}\right) \frac{4D_{22}(\lambda)}{\sqrt{\pi}\sqrt{\frac{\alpha_{01}}{\alpha_{02}}\frac{L_{2m}}{N_{2M}}}\,\mathrm{erfc}\left(\sqrt{\frac{\alpha_{01}}{\alpha_{02}}\frac{N_{2M}}{L_{2m}}}\lambda\right)}$

Además, $\mathcal{E}_2$ *es una función creciente en* λ.

Teorema

Supongamos que (B): $2D_{24}(0) + \left(\frac{\alpha_{01}}{\alpha_{02}} \frac{\overline{A}}{2L_{2m}(R-P)} + 1\right) \frac{4D_{22}(0)}{\sqrt{\pi}\sqrt{\frac{\alpha_{01}}{\alpha_{02}}\frac{L_{2m}}{N_{2M}}}} < 1$ *entonces existe un único* $\overline{\lambda}_2 = \mathcal{E}_2^{-1}(1)$ *tal que para cada* $0 < \lambda < \overline{\lambda}_2$, *el operador* $\mathcal{H}_2$ *tiene un único punto fijo* $g_\lambda \in \mathcal{G}$.

Teorema

Sea $\overline{\lambda} = \min\left(\overline{\lambda}_1, \overline{\lambda}_2\right)$. *Bajo las hipótesis (A) y (B), para cada* $0 < \lambda < \overline{\lambda}$, *el par* (f_λ, g_λ) *satisface las ecuaciones* (1) *y* (2).

La ecuación (3) es equivalente a:

$$\mathcal{W}(\lambda) = 2\lambda, \qquad 0 < \lambda < \overline{\lambda},$$

donde $\mathcal{W}(\lambda) = \mathcal{W}(f_\lambda, g_\lambda, \lambda) = -\text{Ste}_2 \frac{1+\chi_2(g_\lambda)(\infty)}{\phi_2(g_\lambda)(\infty)} + \text{Ste}_1 E_1(f_\lambda)(\lambda) \left(\frac{1+\chi_1(f_\lambda)(\lambda)}{\phi_1(f_\lambda)(\lambda)} - w_1(f_\lambda)(\lambda) \right)$.

Lema

Para cada $\lambda \in (0, \overline{\lambda})$ tenemos $\mathcal{W}_2(\lambda) < \mathcal{W}(\lambda) < \mathcal{W}_1(\lambda)$, donde $\mathcal{W}_1$ y $\mathcal{W}_2$ son funciones continuas dadas por

$$\mathcal{W}_1(\lambda) = \frac{2}{\sqrt{\pi}} \mathrm{Ste}_2 \, \frac{L_{1M}\sqrt{N_{1M}}}{\sqrt{L_{1m}}} \exp\left(\frac{2\mu_{1M}\lambda}{L_{1m}}\right) \frac{1+\frac{\beta_{1M}}{L_{1m}}\lambda^2 \exp\left(\frac{2\mu_{1M}\lambda+N_{1M}\lambda^2}{L_{1m}}\right)}{\operatorname{erf}\left(\sqrt{\frac{N_{1M}}{L_{1m}}}\lambda\right)}, \qquad \lambda > 0,$$

$$\mathcal{W}_2(\lambda) = -\mathrm{Ste}_2 \frac{1+\frac{\alpha_{01}}{\alpha_{02}} \frac{\overline{A}}{2L_{2m}(R-P)} \exp(-(R-P)\lambda^2)}{\frac{\sqrt{\pi}}{2}\sqrt{\frac{\alpha_{01}}{\alpha_{02}}\frac{L_{2m}}{N_{2M}}} \exp\left(\frac{\alpha_{01}}{\alpha_{02}}\frac{N_{2M}}{L_{2m}}\lambda^2\right) \operatorname{erfc}\left(\sqrt{\frac{\alpha_{01}}{\alpha_{02}}\frac{N_{2M}}{L_{2m}}}\lambda\right)}$$

$$+\mathrm{Ste}_1 \exp\left(-\frac{N_{1M}}{L_{1m}}\lambda^2\right) \left[\frac{L_{1m}\sqrt{N_{1m}}\left(1+\frac{\beta_{1m}L_{1m}}{2\mu_{1M}L_{1M}}\left[\frac{\sqrt{\pi}}{2}\sqrt{\frac{L_{1m}}{N_{1M}}}\operatorname{erf}\left(\sqrt{\frac{N_{1M}}{L_{1m}}}\lambda\right) - \lambda \exp\left(-\frac{2\mu_{1M}\lambda+N_{1M}\lambda^2}{L_{1m}}\right)\right]\right)}{\sqrt{L_{1M}} \exp\left(\frac{\mu_{1M}^2 L_{1M}}{L_{1m}^2 N_{1m}}\right)\left[\operatorname{erf}\left(\sqrt{\frac{N_{1m}}{L_{1M}}}\lambda - \sqrt{\frac{L_{1M}}{N_{1m}}}\frac{\mu_{1M}}{L_{1m}}\right) + \operatorname{erf}\left(\sqrt{\frac{L_{1M}}{N_{1m}}}\frac{\mu_{1M}}{L_{1m}}\right)\right]}\right.$$

$$\left. -\beta_{1M}\lambda \exp\left(\frac{N_{1M}}{L_{1m}}\lambda^2\right) \right]$$

que satisfacen $\mathcal{W}_1(0) = \mathcal{W}_2(0) = +\infty$.

Teorema

Si se verifican las hipótesis (A),(B) y además $\mathcal{W}_1(\overline{\lambda}) < 2\overline{\lambda}$ *entonces existe solución* $\widetilde{\lambda}$ *a la ecuación* (3).

Observación

La condición $\mathcal{W}_1(\overline{\lambda}) < 2\overline{\lambda}$ *equivale a*

$$(C): \qquad \ell > \frac{c_{02}(T_m - U_0)w(\overline{\lambda})}{\overline{\lambda}},$$

donde $w(\overline{\lambda}) = \frac{L_{1M}\sqrt{N_{1M}}}{\sqrt{\pi}\sqrt{L_{1m}}} \exp\left(\frac{2\mu_{1M}\overline{\lambda}}{L_{1m}}\right) \frac{1+\frac{\beta_{1M}}{L_{1m}}\overline{\lambda}^2 \exp\left(\frac{2\mu_{1M}\overline{\lambda}+N_{1M}\overline{\lambda}^2}{L_{1m}}\right)}{\operatorname{erf}\left(\sqrt{\frac{N_{1M}}{L_{1m}}}\overline{\lambda}\right)}.$

Teorema

Bajo las hipótesis (A),(B) y (C) existe al menos una solución $(f_{\tilde{\lambda}}, g_{\tilde{\lambda}}, \widetilde{\lambda})$ *al problema (PI) dado por* (1)-(3).

Teorema

Existe solución al problema de Stefan a dos fases dado por

$$T(x,t) = (T^* - T_m) f_{\tilde{\lambda}}\left(\frac{x}{2\sqrt{\alpha_{01} t}}\right) + T_m, \qquad 0 < x \leq s(t), \quad t > 0$$

$$U(x,t) = (T_m - U_0) g_{\tilde{\lambda}}\left(\frac{x}{2\sqrt{\alpha_{01} t}}\right) + T_m, \qquad x \geq s(t), \quad t > 0$$

donde la frontera libre es $s(t) = 2\widetilde{\lambda}\sqrt{\alpha_{01} t}$ *siendo* $(f_{\tilde{\lambda}}, g_{\tilde{\lambda}}, \widetilde{\lambda})$ *solución al problema (PI) dado por* (1)-(3).

Casos particulares

Coeficientes constantes SIN presencia de fuentes de calor [Tu2018]

Datos:

$$\rho_1(T) = \rho_0, \quad c_1(T) = c_{01}, \quad k_1(T) = k_{01}, \quad v_1(T) = \tfrac{\rho_0 c_{01} \mathrm{Pe} \sqrt{\alpha_{01}}}{\sqrt{t}},$$

$$\rho_2(U) = \rho_0, \quad c_2(U) = c_{02}, \quad k_2(U) = k_{02}, \quad v_2(U) = \tfrac{\rho_0 c_{02} \mathrm{Pe} \sqrt{\alpha_{01}}}{\sqrt{t}}$$

$$F_1(T) = F_2(U) = 0$$

Solución:

$$f(\xi) = \frac{\mathrm{erf}(\lambda - \mathrm{Pe}) - \mathrm{erf}(\xi - \mathrm{Pe})}{\mathrm{erf}(\lambda - \mathrm{Pe}) + \mathrm{erf}(\mathrm{Pe})}, \qquad 0 \leq \xi \leq \lambda,$$

$$g(\xi) = \frac{\mathrm{erf}\left(\sqrt{\frac{\alpha_{01}}{\alpha_{02}}}(\lambda - \mathrm{Pe})\right) - \mathrm{erf}\left(\sqrt{\frac{\alpha_{01}}{\alpha_{02}}}(\xi - \mathrm{Pe})\right)}{\mathrm{erfc}\left(\sqrt{\frac{\alpha_{01}}{\alpha_{02}}}(\lambda - \mathrm{Pe})\right)}, \qquad \xi \geq \lambda,$$

$$-\mathrm{Ste}_2 \sqrt{\frac{\alpha_{02}}{\alpha_{01}}} \frac{\exp\left(-\frac{\alpha_{01}}{\alpha_{02}}(\lambda - \mathrm{Pe})^2\right)}{\mathrm{erfc}\left(\sqrt{\frac{\alpha_{01}}{\alpha_{02}}}(\lambda - \mathrm{Pe})\right)} + \mathrm{Ste}_1 \frac{\exp\left(-(\lambda - \mathrm{Pe})^2\right)}{\mathrm{erf}\,(\lambda - \mathrm{Pe}) + \mathrm{erf}(\mathrm{Pe})} = \sqrt{\pi}\lambda.$$

Coeficientes constantes con fuentes exponenciales [BrNa2007]

Datos:

$$F_1(T)(x,t) = \frac{(T^*-T_m)\rho_0 c_{01}}{4t}\exp\left(-\frac{x^2}{4\alpha_{01}t}\right), \qquad F_2(U)(x,t) = \frac{(T_m-U_0)\rho_0 c_{02}}{4t}\exp\left(-\frac{x^2}{2\alpha_{02}t}\right).$$

Solución:

$$\begin{aligned}
f(\xi) &= 1 + \frac{\sqrt{\pi}}{4\mathrm{Pe}}\left\{\exp(\mathrm{Pe}^2)\left[\operatorname{erf}(\xi-\mathrm{Pe})+\operatorname{erf}(\mathrm{Pe})\right]-\operatorname{erf}(\xi)\right\} \\
&\quad - \frac{\operatorname{erf}(\xi-\mathrm{Pe})+\operatorname{erf}(\mathrm{Pe})}{\operatorname{erf}(\lambda-\mathrm{Pe})+\operatorname{erf}(\mathrm{Pe})}\left(1+\frac{\sqrt{\pi}}{4\mathrm{Pe}}\left\{\exp(\mathrm{Pe}^2)\left[\operatorname{erf}(\lambda-\mathrm{Pe})+\operatorname{erf}(\mathrm{Pe})\right]-\operatorname{erf}(\lambda)\right\}\right) \quad 0\le\xi\le\lambda, \\
g(\xi) &= \frac{\pi}{8}\left[\operatorname{erf}^2\left(\sqrt{\frac{\alpha_{01}}{\alpha_{02}}}(\xi-\mathrm{Pe})\right)-\operatorname{erf}^2\left(\sqrt{\frac{\alpha_{01}}{\alpha_{02}}}(\lambda-\mathrm{Pe})\right)\right] \\
&\quad -\frac{\pi}{4}\operatorname{erf}\left(\sqrt{\frac{\alpha_{01}}{\alpha_{02}}}(\lambda-\mathrm{Pe})\right)\left[\operatorname{erf}\left(\sqrt{\frac{\alpha_{01}}{\alpha_{02}}}(\xi-\mathrm{Pe})\right)-\operatorname{erf}\left(\sqrt{\frac{\alpha_{01}}{\alpha_{02}}}(\lambda-\mathrm{Pe})\right)\right] \\
&\quad -\frac{\operatorname{erf}\left(\sqrt{\frac{\alpha_{01}}{\alpha_{02}}}(\xi-\mathrm{Pe})\right)-\operatorname{erf}\left(\sqrt{\frac{\alpha_{01}}{\alpha_{02}}}(\lambda-\mathrm{Pe})\right)}{\operatorname{erfc}\left(\sqrt{\frac{\alpha_{01}}{\alpha_{02}}}(\lambda-\mathrm{Pe})\right)}\left(1+\frac{\pi}{8}\operatorname{erfc}^2\left(\sqrt{\frac{\alpha_{01}}{\alpha_{02}}}(\lambda-\mathrm{Pe})\right)\right) \qquad \xi\ge\lambda,
\end{aligned}$$

$$\begin{aligned}
&\mathrm{Ste}_1\exp(2\mathrm{Pe}\lambda-\lambda^2)\left(\frac{1+\frac{\sqrt{\pi}}{4\mathrm{Pe}}\left\{\exp(\mathrm{Pe}^2)\left[\operatorname{erf}(\lambda-\mathrm{Pe})+\operatorname{erf}(\mathrm{Pe})\right]-\operatorname{erf}(\lambda)\right\}}{\frac{\sqrt{\pi}}{2}\exp(\mathrm{Pe}^2)\left(\operatorname{erf}(\lambda-\mathrm{Pe})+\operatorname{erf}(\mathrm{Pe})\right)}-\frac{1-\exp(-2\mathrm{Pe}\lambda)}{2\mathrm{Pe}}\right) \\
&-\mathrm{Ste}_2\frac{\left(1+\frac{\pi}{8}\operatorname{erfc}^2\left(\sqrt{\frac{\alpha_{01}}{\alpha_{02}}}(\lambda-\mathrm{Pe})\right)\right)}{\frac{\sqrt{\pi}}{2}\sqrt{\frac{\alpha_{01}}{\alpha_{02}}}\exp\left(\frac{\alpha_{01}}{\alpha_{02}}(\lambda-\mathrm{Pe})^2\right)\operatorname{erfc}\left(\sqrt{\frac{\alpha_{01}}{\alpha_{02}}}(\lambda-\mathrm{Pe})\right)} = 2\lambda.
\end{aligned}$$

Conclusiones

- Se planteó un problema (P) de Stefan a dos fases no clásico con coeficientes térmicos dependientes de la temperatura
- Se transformó el problema de Stefan en un problema diferencial ordinario (PDO)
- Se estudió un problema equivalente (PI) dado por dos ecuaciones integrales acopladas a una condición extra para el coeficiente de la frontera libre, esto es (1)-(3).
- Se demostró existencia de solución al problema (PI).
- Se estudiaron algunos casos particulares.

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