Topological Frobenius reciprocity and invarian Hermitian forms

Expositor: Tim Bratten (UNICEN, bratten@exa.unicen.edu.ar)
Autor/es: Tim Bratten (UNICEN, bratten@exa.unicen.edu.ar); Mauro Natale (UNICEN, mauro_natale@hotmail.com)

Motivated by a question about Frobenius reciprocity, raised by D. Vogan in [3, Question 10.2], we prove a topological Frobenius reciprocity and use it to study invariant Hermitian forms on representations. In particular, let G_0 be a real reductive group of Harish-Chandra class with complexified Lie algebra $\mathfrak{g}$. Suppose $\mathfrak{p} \subseteq \mathfrak{g}$ is a *nice* parabolic subalgebra [3]. Then the normalizer, L_0, of $\mathfrak{p}$ in G_0 is a Levi subgroup. Let S be the orbit of $\mathfrak{p}$ in the corresponding generalized flag manifold and let q be the vanishing number for S. Suppose $V_{\min}$ is a minimal globalization for L_0 with regular, antidominant infinitesimal character and let $\mathfrak{u}$ be the nilradical of $\mathfrak{p}$. *Topological Frobenius reciprocity* is the natural correspondence

$$\mathrm{Hom}_{G_0}\left(H_c^q(S, \mathcal{O}(\mathfrak{p}, V_{\min})), M_{\max}\right) \cong \mathrm{Hom}_{L_0}\left(V_{\min}, H_q(\mathfrak{u}, M_{\max})\right)$$

where $H_c^q\left(S, \mathcal{O}(\mathfrak{p}, V_{\min})\right)$ is the representation for G_0 on the compactly supported cohomology of the analytic sheaf $\mathcal{O}(\mathfrak{p}, V_{\min})$, $M_{\max}$ is a quasisimple maximal globalization for G_0 and $H_q(\mathfrak{u}, M_{\max})$ is the representation for L_0 on the Lie algebra homology group. The reciprocity result follows from the geometric description of $H_c^q\left(S, \mathcal{O}(\mathfrak{p}, V_{\min})\right)$ and properties of the minimal and maximal globalization with respect to Lie algebra homology [1], [2]. We apply the topological Frobenius reciprocity to study the invariant Hermitian forms on $H_c^q\left(S, \mathcal{O}(\mathfrak{p}, V_{\min})\right)$. Put $I_{\min} = H_c^q\left(S, \mathcal{O}(\mathfrak{p}, V_{\min})\right)$, let $\chi_{\mathfrak{u}}$ be the determinant character for the L_0-action on $\mathfrak{u}$ and let $Z(\mathfrak{l})$ be the center of the enveloping algebra of the complexified Lie algebra, $\mathfrak{l}$, of L_0. It turns out one would like to know that $V_{\min}$ is isomorphic to the corresponding $Z(\mathfrak{l})$-eigenspace in the representation $H_s(\mathfrak{u}^{\mathrm{op}}, I_{\min}) \otimes \chi_{\mathfrak{u}}$, where $s = \dim_{\mathbb{C}}(S)$ and $\mathfrak{u}^{\mathrm{op}}$ is the nilradical of the parabolic subalgebra, $\mathfrak{p}^{\mathrm{op}}$, opposite to $\mathfrak{p}$. To establish this last result we generalize a geometric duality theorem established by M. Zabcic for discrete series in his Ph.D thesis [4]. In this way we obtain a natural correspondence between invariant Hermitian forms on $H_c^q\left(S, \mathcal{O}(\mathfrak{p}, V_{\min})\right)$ and invariant Hermitian forms on $V_{\min}$.

[1] T. Bratten: *A comparison theorem for Lie algebra homology groups.* Pac. J. of Math., **182** (1), (1998), 23-36.
[2] T. Bratten: *A simple proof of the algebraic version of a conjecture by Vogan.* J. Lie Theory, **18** (1), (2008), 83-93.
[3] D. Vogan: *Unitary representations and complex analysis.* In *Representation Theory and Complex Analysis*, Lecture Notes in Mathematics, **1931**. Springer, Berlin, Heidelberg (2008), 259-344.
[4] M. Zabcic: *Geometry of discrete series.* Ph.D. Thesis, University of Utah, 1988.