Hemi-Nelson Algebras

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This work is motivated by the common features shared by several algebraic structures associated with intuitionistic logics, including Heyting algebras, semi-Heyting algebras, subresiduated lattices, Nelson algebras, semi-Nelson algebras, and subresiduated Nelson algebras. Our aim is to provide a unified framework encompassing these varieties.

Hemi-implicative lattices [2] are defined as lattices with a greatest element, denoted by 1, equipped with a binary operation $\to$ satisfying the equation $x \to x = 1$ and the inequality $x \wedge (x \to y) \leq y$. The latter inequality can be replaced by the following quasi-equation:

$$\text{If } z \leq x \to y, \text{ then } z \wedge x \leq y.$$

A bounded distributive hemi-implicative lattice is a hemi-implicative lattice whose underlying lattice is distributive and has a greatest element 1. Many varieties of interest in algebraic logic are subvarieties of the variety of bounded distributive hemi-implicative lattices. Relevant examples include Heyting algebras, residuated lattices, RWH-algebras [3], semi-Heyting algebras, and bounded distributive Hilbert lattices (that is, Hilbert algebras [1,4] whose natural order determines a bounded distributive lattice).

Nelson's constructive logic with strong negation, introduced and studied in [5] (see also [6,7,8]), is a non-classical logic that combines the constructive approach of positive intuitionistic logic with a classical (that is, De Morgan) negation. The algebraic semantics of this logic are given by Nelson algebras. The class of Nelson algebras, which forms a variety, has been studied since the late 1950s (initially by Rasiowa; see [6]). At the end of the 1970s, Fidel and Vakarelov independently proved that every Nelson algebra can be represented as a particular binary product (later called a *twist structure*) of a Heyting algebra.

The main goal of this talk is to extend this twist construction to the setting of bounded distributive hemi-implicative lattices, thereby obtaining a new variety whose members we call *hemi-Nelson algebras*. More precisely, an algebra $\langle T, \wedge, \vee, \to, \sim, 0, 1\rangle$ of type $(2,2,2,1,0,0)$ is called a hemi-Nelson algebra (or an *h-Nelson algebra*) if $\langle T, \wedge, \vee, \sim, 0, 1\rangle$ is a Kleene algebra and the following conditions hold for every $x, y, z \in T$:

1. $x \to x = 1$,
2. $x \wedge (x \to y) \leq x \wedge (\sim x \vee y)$,
3. $\sim (x \to y) \to (x \wedge \sim y) = 1$,
4. $(x \wedge \sim y) \to \sim (x \to y) = 1$,
5. $(x \wedge y \wedge (x \to y)) \to (x \wedge (x \to y)) = 1$,
6. $(x \wedge (x \to y)) \to (x \wedge y \wedge (x \to y)) = 1$,
7. $(x \to y) \vee (z \wedge \sim z) = (x \vee (z \wedge \sim z)) \to (y \vee (z \wedge \sim z))$.

We show that every hemi-Nelson algebra can be represented as a twist structure over a bounded distributive hemi-implicative lattice. We also prove that, for every hemi-Nelson algebra, there exists an order isomorphism between its congruence lattice and the lattice of its h-implicative filters (which constitute a particular class of implicative filters). Furthermore, we establish an equivalence between the algebraic category of bounded distributive hemi-implicative lattices and the algebraic category of centered hemi-Nelson algebras, where a centered hemi-Nelson algebra is defined as a hemi-Nelson algebra together with a center, that is, an element fixed by the involution (such an element is necessarily unique). Finally, we apply some of the developed results to investigate several aspects of semi-Nelson algebras and subresiduated Nelson algebras.

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