Schauder estimates for flat solutions of fully nonlinear elliptic PDEs with Dini Data

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Abstract:

In this talk, we establish local Schauder estimates for flat viscosity solutions, that is, solutions with sufficiently small norms, to a class of fully nonlinear elliptic partial differential equations of the form

$$F(D^2u, x) + \langle \mathfrak{B}(x), Du\rangle = f(x) \quad \text{in} \quad \text{B\_1} \subset \mathbb{R}^n, \tag{1}$$

where the operator F is differentiable, though not necessarily convex or concave. In addition, we impose suitable Dini-type continuity assumptions on the data. Our methodology is based on geometric tangential techniques, combined with compactness and perturbative arguments.

The results presented in this talk are part of my master's dissertation and are based on joint work with my advisor, Dr. João Vitor da Silva, and Dr. Junior da Silva Bessa. A detailed account of these results has been published in the Bulletin of the Brazilian Mathematical Society (see [1]).

Keywords: Fully nonlinear elliptic PDEs, Schauder estimates, Dini continuity.

Introduction:

Caffarelli (see [[2],Theorem 8.1]) established Schauder estimates for the inhomogeneous problem $F(D^2u, x) = f(x)$ via a perturbation and compactness technique under suitable Hölder assumptions on data, and concavity or convexity assumption on the operator F.

For nearly twenty years, the question of whether *arbitrary* fully nonlinear elliptic equations $F(D^2\mathfrak{h}) = 0$ in B_1 (resp. $F(D^2\mathfrak{h}) = f(x)$) admit a general C^2 *a priori* regularity theory remained unresolved. This problem was finally settled by Nadirashvili and Vlăduţ's counterexamples to $C^{1,1}$ regularity in [5], which concluded this line of investigation. Their works, however, stimulated new research directions. In light of the inherent obstacles to developing a universal existence theory for classical solutions to fully nonlinear equations, a major focus of current research involves identifying supplementary structural or qualitative conditions on $F(\mathrm{X}, \cdot), F(\cdot, x), f$, and even on u that enable C^2 estimates.

Main Results:

We make the following structural assumptions:

[A1] **Uniform Ellipticity:** The operator $F : \mathrm{Sym}(n) \times \Omega \to \mathbb{R}$ is fully nonlinear and uniformly elliptic, with ellipticity constants $0 < \lambda \le \Lambda$. Specifically, we require

$$P^{-}\_\lambda, \Lambda(\mathrm{N}) \le F(\mathrm{M}+\mathrm{N}, x) - F(\mathrm{M}, x) \le P^{+}\_\lambda, \Lambda(\mathrm{N}), \ \forall\, x \in \Omega, \ \forall\, \mathrm{M}, \mathrm{N} \in \mathrm{Sym}(n), \ \ \text{whith}, \ \ \mathrm{N} \ge 0. \tag{2}$$

[A2] **Differentiability of the Nonlinearity:** We assume that $F \in C^1(\mathrm{Sym}(n))$, and there exists a modulus of continuity $\omega : [0, \infty) \to [0, \infty)$ such that

$$\|\mathrm{D_M}F(\mathrm{X}, x) - \mathrm{D_M}F(\mathrm{Y}, x)\| \le \omega(\|\mathrm{X} - \mathrm{Y}\|), \quad \forall\, x \in \Omega \text{ and } \forall\, \mathrm{X}, \mathrm{Y} \in \mathrm{Sym}(n).$$

[A3] **Dini continuity in the L^n-sense:** There exists a modulus of continuity $\tau : [0, \infty) \to [0, \infty)$, and non-negative constants $\mathrm{C}_f, \mathrm{C}_{\theta_F}, \mathrm{C}_{\mathfrak{B}}$, such that

$$\left(\frac{1}{|\mathrm{B}_r|}\int_{\mathrm{B}_r(x_0)}|f(x)-f(x_0)|^{p_0}dx\right)^{\frac{1}{p_0}} \leq \mathrm{C}_f\tau(r), \quad \left(\frac{1}{|\mathrm{B}_r|}\int_{\mathrm{B}_r(x_0)}|\theta_F(x,x_0)|^{p_0}dx\right)^{\frac{1}{p_0}} \leq \mathrm{C}_{\theta_F}\tau(r),$$

$$\left(\frac{1}{|\mathrm{B}_r|}\int_{\mathrm{B}_r(x_0)}|\mathfrak{B}(x)-\mathfrak{B}(x_0)|^{p_0}dx\right)^{\frac{1}{p_0}} \leq \mathrm{C}_{\mathfrak{B}}\tau(r)$$

where τ satisfies the Dini condition $\displaystyle\int_0^1 \frac{\tau(s)}{s}ds < \infty$ and $p_0 > n$.

[A4] **Compatibility condition:** We further suppose the **nullity conditions**:

$$\liminf_{s\to 0^+} \sup_{0<r\leq\frac{1}{2}} \frac{\tau(rs)}{\tau(r)} = 0 \quad \text{and} \quad \liminf_{s\to 0^+} \sup_{k\in\mathbb{N}} \frac{s^{\alpha_0}\tau(s^k)}{\tau(s^{k+1})} = 0 \text{ for some } \alpha_0 \in (0,1].$$

Within this framework, we are able to derive the following local Schauder estimates.

Teorema (Local $C^{2,\mathbf{Dini}}$ regularity):

Let $u \in C^0(\mathrm{B}\_1)$ be a viscosity solution to (1),

where the assumptions (A1)–(A4) are in force. There exists $\delta\_0 > 0$, depending only upon n, λ, Λ, ω, and $\tau(1)$, such that if

$$\|u\|_{L^\infty(\mathrm{B}_1)} \leq \delta_0,$$

then $u \in C\_\mathrm{loc}^{2,\psi}(\mathrm{B}\_1)$ and

$$\|u\|_{C^{2,\psi}(\mathrm{B}_{1/2})} \leq \mathrm{C}_0 \cdot \delta_0,$$

where C_0 > 0 depends only upon n, λ, Λ, α_0, ω, and τ and $\psi(t) = \tau(t) + \displaystyle\int \_0^t \frac{\tau(s)}{s}ds.$

Our results can be viewed as an extension of the work by dos Prazeres and Teixeira [[3], Theorem 2.2] and Kovats [[4], Theorem 1], now within the framework of linear drift terms, non-convexity, and Dini continuity assumptions.

Trabajo en conjunto con Dr. João Vitor da Silva, (IMECC/UNICAMP) y Dr. Junior da Silva Bessa, (IMECC/UNICAMP).

Referencias

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