# Interior and Flat-Boundary $C^{1,\alpha}$ Regularity for $p$-Poisson-Type Equations in the Plane

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In this work, we study the Hölder regularity of the gradient of weak solutions to degenerate quasilinear elliptic equations in dimension two, both in the interior of the domain and up to a flat portion of the boundary.

In the interior setting, let $p > 2$, $q > 2$, and $0 < \tau \leq 1$. We consider a weak solution $u \in W^{1,p}(B\_1)$ of the equation

$$-\operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big) = f + \operatorname{div} F \qquad \text{in } B_1 \subset \mathbb{R}^2,$$

under the assumptions

$$f \in L^q(B_1), \qquad F \in C^{0,\tau}(B_1;\mathbb{R}^2).$$

Defining

$$\alpha_{\text{int}} := \text{mín}\left\{ \frac{1-\frac{2}{q}}{p-1}, \frac{\tau}{p-1} \right\},$$

we establish that

$$u \in C^{1,\alpha_{\text{int}}}_{\text{loc}}(B_1)$$

and obtain the quantitative estimate

$$\|u\|_{C^{1,\alpha_{\text{int}}}(B_{1/2})} \leq C \left( \|u\|_{L^\infty(B_1)} + \|f\|_{L^q(B_1)}^{\frac{1}{p-1}} + [F]\_C^{0,\tau}(B\_1)^{\frac{1}{p-1}} \right).$$

The exponent $\alpha\_\text{int}$ highlights the interaction between the integrability of the source term $f$, the Hölder regularity of the vector field $F$, and the intrinsic degeneracy of the $p$-Laplacian operator.

We next address regularity up to the flat boundary. Denoting

$$B_1^+ := B_1 \cap \{x_2 > 0\}, \qquad B_1' := B_1 \cap \{x_2 = 0\},$$

we consider the Dirichlet problem

$$\begin{cases} -\operatorname{div}(\mathcal{A}(\nabla u)) = f + \operatorname{div} F & \text{in } B_1^+, \\ u = g\&\text{on } B\_1', \end{cases}$$

where $\mathcal{A}$ is an autonomous vector field with $p$-Laplacian-type structure, satisfying the standard growth, coercivity, and $p$-elliptic monotonicity conditions. Assuming

$$f \in L^q(B_1^+), \qquad F \in C^{0,\theta}(\overline{B_1^+};\mathbb{R}^2), \qquad g \in C^{1,\gamma}(B_1'),$$

with $0 < \theta < 1$ and $0 < \gamma \leq 1$, we prove that, for every

$$0 < \alpha_\partial < \text{mín}\left\{ \frac{1-\frac{2}{q}}{p-1}, \frac{\theta}{p-1}, \gamma \right\},$$

one has

$$u \in C^{1,\alpha_\partial}(\overline{B_{1/2}^+}),$$

together with the estimate

$$[\nabla u]\_C^{0,\alpha\_\partial}(\overline{B\_1/2^+}) \leq C \left( \|\nabla u\|\_L^\infty(B\_1^+) + \|g\|\_C^{1,\gamma}(B\_1') + \|f\|\_L^q(B\_1^+)^{\frac{1}{p-1}} + [F]\_C^{0,\theta}(\overline{B\_1^+})^{\frac{1}{p-1}} \right).$$

Consequently, the gradient admits a Hölder continuous extension up to the flat boundary, with an exponent determined by the regularities of the terms $f$, $F$, and the Dirichlet datum $g$.

The proofs combine perturbative *a priori* estimates with rescaling, compactness, and *blow-up* arguments. The analysis of the limiting profiles is completed by means of Liouville-type principles, while a regularization and approximation procedure allows the estimates obtained to be transferred to the setting of weak solutions.

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