Sub-Riemannian and sub-Lorentzian geodesics of the oscillator groups

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The oscillator groups are four-dimensional solvable Lie groups, extensions of the Heisenberg Lie group. We present them as circle bundles of the standard contact sub-Riemannian structure on $\mathbb{R}^3$. We define sub-Riemannian and sub-Lorentzian structures on them, describe their geodesics explicitly and determine which of them are periodic.

We also study a generalization: Let G be a semisimple Lie group and K a compact subgroup such that (G, K) is a symmetric pair of the compact or the noncompact type. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K, respectively. We consider on $\mathfrak{g}$ the usual nilpotent Lie group structure with center $\mathfrak{k}$ and call it $N(G,K)$. A suitable semidirect product with $\mathrm{Ad}(K)$ yields a solvable Lie group $\mathrm{Osc}(G,K)$, which generalizes the oscillator groups and is also quadratic (that is, it possesses bi-invariant metrics; in particular, a canonical one). We define nonholonomic pseudo-Riemannian structures on it and find their geodesics explicitly. In doing so, we obtain a result that may be not merely auxiliary: formulas for the monoparametric subgroups of $\mathrm{Osc}(G,K)$ and for the sub-Riemannian geodesics of $N(G,K)$ with the standard left-invariant distribution $\mathcal{D}$. Moreover, when G/K is compact with rank one, we realize $\mathrm{Osc}(G,K)$ as a Stiefel manifold of $(N(G,K), \mathcal{D})$, up to finite coverings.

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