New integral inequalities of the Hermite-Hadamard type for functions (h, m)-convex twice differentiable

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Convex functions have been generalized widely; highlighting the m-convex function, r-convex function, h-convex function, (h, m)-convex function, s-convex function and many others. Readers interested in going through many of these extensions and generalizations of the classical notion of convexity can consult [1].

For convex functions, the following inequality is known, undoubtedly one of the most famous in Mathematics, for its multiple connections and applications:

$$\phi\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b \phi(x)\,dx \leq \frac{\phi(a)+\phi(b)}{2},$$

this is called the Hermite–Hadamard inequality.

This inequality was published by Hermite in 1883 ([2]) and independently by Hadamard in 1893 ([3]). In the last 30 years especially, many researchers have focused their attention on this inequality and many results have appeared.

In [4] we presented the following definitions.

Let $h : [0, 1] \to \mathbb{R}$ be a nonnegative function, $h \neq 0$ and $\psi : I = [0, +\infty) \to [0, +\infty)$. If inequality

$$\psi\left(\tau\xi + m(1-\tau)\varsigma\right) \leq h^s(\tau)\psi(\xi) + m(1-h^s(\tau))\psi\left(\varsigma\right)$$

is fulfilled for all $\xi, \varsigma \in I$ and $\tau \in [0, 1]$, where $m \in [0, 1]$, $s \in [-1, 1]$. Then a function ψ is called a (h, m)-convex modified of the first type on I.

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is fulfilled for all $\xi, \varsigma \in I$ and $\tau \in [0, 1]$, where $m \in [0, 1]$, $s \in [-1, 1]$. Then a function ψ is called a (h, m)-convex modified of the second type on I.

Remark. The reader can verify, without much difficulty, that various functional classes are particular cases of these definitions, for the appropriate choice of h, m, s.

Next we present the weighted integral operators, which will be the basis of our work.

Let $\phi \in L([a, b])$ and let w be a continuous and positive function, $w : [0, 1] \to [0, +\infty)$, with second order derivatives integrable on I. Then the weighted fractional integrals are defined by (right and left respectively):

$$J^w_{a^+}\phi(r) = \int_a^b w''\left(\frac{\sigma-a}{b-a}\right)\phi(\sigma)d\sigma$$

and

$$J^w_{b^-}\phi(r) = \int_a^b w''\left(\frac{b-\sigma}{b-a}\right)\phi(\sigma)d\sigma.$$

In this work, we obtain different variants of the Hermite-Hadamard inequality, in the framework of the (h, m)-convex modified functions, via generalized operators of the Definitions presented before.

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