Generalized porosity and Muckenhoupt weights

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In [2], the authors give a characterization of those sets E in the euclidean space $\mathbb{R}^n$ for which the nonnegative function $w(x) := \operatorname{dist}(x, E)^{-\alpha}$ belongs to the Muckenhoupt class $A_1(\mathbb{R}^n)$ (see [3]). The necessary geometrical property of these kind of sets is known as (weak) porosity and includes lower dimensional sets as those considered in [1]. Our search intends to extend and compare porosity as a property related to Muckenhoupt classes in more general geometric settings.

Our first approach, based on dyadic analysis, deals with the cases of parabolic metrics that generalize the basic planar case, where the space $\mathbb{R}^2$ is equipped with the translation invariant parabolic metric $d_p(x, y) = \sup\{|x_1 - y_1|, \sqrt{|x_2 - y_2|}\}$, with $x = (x_1, x_2)$ and $y = (y_1, y_2)$. The space $(\mathbb{R}^2, d_p, m)$, where m is the Lebesgue measure, is 3-Ahlfors regular. In fact, $m(B_p(x, r)) = 4r^3$ for every $x \in \mathbb{R}^2$ and every $r > 0$. Our main results so far are, the extension of this notion of porosity to "parabolic porosity" and the proof of the equivalence of the parabolic porosity of E with the $A_1^p(\mathbb{R}^2)$ condition for $\operatorname{dist}_p(\cdot, E)^{-\alpha}$, i.e.,

$$\frac{1}{4r^3}\int_{B_p(x,r)} \frac{dy}{d_p(y, E)^{-\alpha}} \leq \frac{C}{d_p(x, E)^{-\alpha}},$$

for every $x \in \mathbb{R}^2$, every $r > 0$ and some $C > 0$. We also construct examples of parabolic porous sets that are not euclidean porous sets and conversely. We then extend the above results to more general parabolic metrics in $\mathbb{R}^n$ and finally, for the general setting in metric measure spaces, we are able to prove the following result.

Theorem. Let (X, d, μ) be a complete metric space with a complete doubling measure and let E be a non empty subset of X. We have that if there exists $\alpha > 0$ such that $d(x, E)^{-\alpha}$ belongs to $A_1(X, d, \mu)$, then there exist constants $c, \delta \in (0, 1)$ such that for every ball B in (X, d), there exist $x_1, ..., x_N \in B$ as well as $r_1, ..., r_N > 0$ (for some $N = N(B) \in \mathbb{N}$) satisfying (i) $\gamma(x_j, B) > r_j > 0$; (ii) the balls $B(x_j, r_j)$ are pairwise disjoint; (iii) $r_j \geq \delta\gamma(B)$ and (iv) $\sum_{j=1}^{n} \mu(B(x_j, r_j)) \geq c\mu(B)$, where, for $y \in B$, $\gamma(y, B) := \sup\{r > 0 : B(y, r) \subset B \setminus E\}$ and $\gamma(B) := \sup_{y\in B} \gamma(y, B)$.

Let us mention that in the preprint [4], posted on Arxiv Math by the end of June 2023, a generalization of the result obtained in [2] is stated under some "annular decay property" assumption on the considered spaces.

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Referencias

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