A construction of a λ- Poisson generic sequence

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Years ago Zeev Rudnick defined the λ-Poisson generic sequences as the infinite sequences of symbols in a finite alphabet where the number of occurrences of long words in the initial segments follow the Poisson distribution with parameter λ. Benjamin Weiss and Yuval Peres [5] proved that almost all sequences with respect to Lebesgue measure are Poisson generic (see also [1]). Despite this result, no explicit example has yet been given.

In this talk I will present a construction of an explicit λ-Poisson generic sequence over an alphabet of at least three symbols, for any fixed positive real number λ. Since λ-Poisson genericity implies Borel normality, the constructed sequence is Borel normal. The same construction provides explicit instances of Borel normal sequences that are not λ-Poisson generic.

Given $x \in \Omega^{\mathbb{N}}$, and a positive real number λ, $i \in \mathbb{N}_0$ and $k \in \mathbb{N}$ we write $Z^{\lambda}_{i,k}(x)$ for the proportion of words of length k that occur exactly i times in the prefix of x of length $\lfloor \lambda b^k \rfloor$.

Definition 1

Let λ be a positive real number. A sequence $x \in \Omega^{\mathbb{N}}$ is λ-Poisson generic if for every $i \in \mathbb{N}_0$,

$$\lim_{k\to\infty} Z^{\lambda}_{i,k}(x) = e^{-\lambda}\frac{\lambda^i}{i!}.$$

A sequence is Poisson generic if it is λ-Poisson generic for all positive real numbers λ.

The main result I will discuss is the following:

Theorem 1 ([Becher and S.H. [3, Theorem 1])

Let λ be a positive real number and Ω a b-symbol alphabet, $b \geq 3$. Let $(p_i)_{i\in\mathbb{N}_0}$ be a sequence of non-negative real numbers such that $\sum\limits_{i\geq 0} p_i = 1$ and $\sum\limits_{i\geq 0} ip_i = \lambda$. Then there is a construction of an infinite sequence x over alphabet Ω, which satisfies for every $i \in \mathbb{N}_0$,

$$\lim_{k\to\infty} Z^{\lambda}_{i,k}(x) = p_i.$$

The construction in Theorem 1 consists in concatenating segments of any infinite de Bruijn sequence [2, Theorem 1], which is a sequence that satisfies that each initial segment of length b^k is a cyclic de Bruijn sequence of order k. Our construction works by selecting segments of this given sequence and repeating them as many times as determined by the probabilities p_i, for every $i \in \mathbb{N}_0$.

Weiss showed that 1-Poisson genericity implies Borel normality and that the two notions do not coincide [6], witnessed by the Champernowne sequence [4]. The next result is a normality criterion that generalizes this fact. In contrast to Theorem 1, this result does not require the alphabet size b to be greater than 2.

Theorem 2 ([Becher and S.H. [3, Theorem 2])

Let Ω be a b-symbol alphabet, $b \geq 2$, and let $x \in \Omega^{\mathbb{N}}$. We fix a positive real number λ and define for every $i \in \mathbb{N}_0$ the numbers $p_i = \liminf_{k\to\infty} Z^{\lambda}_{i,k}(x)$. If the numbers p_i satisfy $\sum_{i\geq 0} ip_i = \lambda$ then x is normal to base b.

As a consequence of Theorem 2 we obtain the following:

Corollary 1

Every λ-Poisson generic sequence is Borel normal, but the two notions do not coincide. The construction in Theorem 1 yields infinitely many Borel normal sequences which are not λ-Poisson generic.

Trabajo en conjunto con Verónica Becher (Universidad de Buenos Aires).

Referencias

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